Laudatio zur Ehrenpromotion von Prof. Dr. Albert Shiryaev

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Dear colleagues,

dear students,

dear Albert Nikolaevich,

the first occasion where I listened to a talk given by Albert Shiryaev was in an impressive environment, namely in the Finlandia Hall in Helsinki. He was a plenary speaker at the International Congress of Mathematicians in 1978 which was organized in such a wonderful way by our Finnish colleagues. Contrary to the music events which are usually held in this concert hall, for this event the price of the seats was independent of the Euclidean distance to those acting on stage. So I took my chance. I was sitting in the first row listing to the concert of Mathematics given there. At that time I would not have dared to predict that in the 23 years to come there would be many occasions to listen to talks given by this speaker and that my future colleagues in Freiburg and I would have the privilege to discuss mathematics with him during many hours. The theme he was interpreting on stage was *Absolute continuity and singularity of probability measures in functional spaces.*

There is a classical result due to Kakutani (1948).

Absolute continuity/Singularity Alternative

Kakutani 1948

$(\mu_n)_{n\geq 1}$, $(\widetilde{\mu}_n)_{n\geq 1}$ sequences of measures such that $\widetilde{\mu}_n \ll \mu_n \quad (n \geq 1)$

Define $P = \mu_1 \times \mu_2 \times \ldots$, $\widetilde{P} = \widetilde{\mu}_1 \times \widetilde{\mu}_2 \times \ldots$

$\Longrightarrow \quad \widetilde{P} \ll P \quad \text{or} \quad \widetilde{P} \perp P$

Series of papers with Yuri Kabanov, Robert Liptser (1977, 1978)

General approach to ACS

P and $\widetilde{P}$ be two probability measures on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$

Define $P_t = P/\mathcal{F}_t$, $\widetilde{P}_t = \widetilde{P}/\mathcal{F}_t$ and $\widetilde{P} \overset{\text{loc}}{\ll} P$, if
$\widetilde{P}_t \ll P_t \ (t \geq 0)$ (local absolute continuity)

Theorem Assume $\widetilde{P} \overset{\text{loc}}{\ll} P$, then

$$\widetilde{P} \ll P \quad \Longleftrightarrow \quad \widetilde{P}(B_\infty(M) < \infty) = 1$$
$$\widetilde{P} \perp P \quad \Longleftrightarrow \quad \widetilde{P}(B_\infty(M) = \infty) = 1$$

Many variations of this in discrete and continuous time

$\longrightarrow$ Predictable criteria

$$Z_t = \frac{d\widetilde{P}_t}{dP_t}$$

$$M_t = \int\limits_0^t Z_{s-}^{\oplus} dZ_s$$

$$B_t(M) \ = \ \langle M^c \rangle_t + \int\limits_0^t \frac{x^2}{1+|x|}\,\nu(ds,dx)$$

Now called Hellinger process

Explanation of this terminology:

can be interpreted as Hellinger distance between local characteristics

One can distinguish several periods in the scientific, pedagogical and publishing activity of Albert Shiryaev. I have tried to draw a map of the topics where he made significant contributions. They look rather disjoint but actually are inseparably interlaced if one looks closer at his results.

The notion of predictability is such a link.

Signal Detection
Optimal Stopping

Nonlinear Filtering
Sequential Analysis

Spectral theory of
stationary processes

Functional limit
theorems

Absolute continuity
Singularity

Semimartingale
theory

Mathematical Finance

Signal detection and optimal stopping

Observe a process (X_t) given by

$$dX_t = r\theta_t dt + \sigma dW_t$$

where $\theta_t = 1_{\{t \geq \theta\}} \qquad \theta \sim \exp(\lambda)$

We want to detect the time where a signal appears

Suppose τ is the 'time of alarm'

Natural formulation of the mathematical problem

$$R^* = \inf_{\tau} \left\{ P[\tau < \theta] + cE[\max(0, \tau - \theta)] \right\}$$

Find optimal stopping time

Shiryaev showed that it is sufficient to consider the 'sufficient statistic'

$$\pi_t = P\left[\theta \leq t \mid \mathcal{F}_t^X\right] = E\left[\theta_t \mid \mathcal{F}_t^X\right]$$

He derived the stochastic differential equation

$$d\pi_t \;=\; \lambda(1 - \pi_t)dt \;+\; \frac{r}{\sigma}\pi_t(1 - \pi_t)d\overline{W}_t$$

where $\overline{W}_t \;=\; \frac{1}{\sigma}X_t \;-\; \frac{r}{\sigma}\int_0^t \pi_s ds$ is the innovation process

$\longrightarrow$ first **nonlinear** filter for estimation problem

Limit theorems for semimartingales

$$X_t = V_t + M_t \qquad (M_t) \in \mathcal{M}_{\text{loc}}, \;\; (V_t) \in \mathcal{V}_{\text{loc}}$$

Canonical representation of $X = (X_t)$

$$X_t = B_t + M_t^c + \int_0^t \int_{|x|\le 1} x(\mu - \nu)(ds, dx) + \int_0^t \int_{|x|>1} x\mu(ds, dx)$$

triplet of local characteristics (predictable)

$(B, \langle M^c \rangle, \nu)$

Consider sequence of semimartingales (X^n)

Question: Convergence of $(X^n)_{n\ge 1}$?

X^n may be considered as a $D(\mathbb{R})$-value random variable

$D(\mathbb{R})$ with Skorohod topology

$$X^n \xrightarrow{\mathcal{D}} M$$

Classical case: $X_t^n = \sum_{k=0}^{[nt]} \xi_{nk}$ partial sum process

$$X^n \xrightarrow{\mathcal{D}} W \qquad (W_t) \;\; \text{Brownian motion}$$

Theorem (Liptser, Shiryaev (1980))

Let (X_t^n) be semimartingales with triplets $(B^n, \langle X^{nc}\rangle, \nu^n)$ and let (M_t) be a continuous Gaussian martingale. Suppose for all $t > 0$ and $\varepsilon > 0$

(A) $$\int_0^t \int_{|x|>\varepsilon} \nu^n(ds, dx) \xrightarrow{P} 0$$

(B) $$\sup_{0<s\leq t} \left| B^{nc} + \sum_{0<s\leq t} \int_{|x|\leq\varepsilon} x\nu^n(\{s\}, dx) \right| \xrightarrow{P} 0$$

(C) $$\langle X^{nc}\rangle_t + \int_0^t \int_{|x|\leq\varepsilon} x^2\nu^n(ds, dx) - \sum_{0<s\leq t} \left(\int_{|x|\leq\varepsilon} x\nu^n(\{s\}, dx) \right)^2 \xrightarrow{P} \langle M\rangle_t$$

Then $X^n \xrightarrow{\mathcal{D}} M$

Quantitative versions: rates of convergence

Important aspect: Synthesis of two areas which had been disjoint before

(1) Functional limit theorems (asymptotics of distributions) based on topological-analytical concepts

(2) Semimartingale theory (stochastic analysis) based on probabilistic concepts

common object: space of trajectories

Skorohodspace D $\longleftrightarrow$ càdlàg paths

Starting point for much more general theory

Given a sequence of semimartingales (X_t^n) with triplets of local characteristics (B^n, C^n, ν^n)

Idea:
$$B^n \longrightarrow B$$
$$C^n \longrightarrow C$$
$$\nu^n \longrightarrow \nu$$

$\Longrightarrow X^n \xrightarrow{\mathcal{D}} X$ with characteristics (B, C, ν)

identification of the limit (X_t)

Theme of the book with Jean Jacod
Limit Theorems for Stochastic Processes (Springer 1987)

many original results

Convergence of stochastic intergrals

$\left(\int H^n dX^n\right)_t$ H^n predictable processes

$$\int H^n dX^n \xrightarrow{\mathcal{D}} \int H dX$$

Application in mathematical finance

The Russian Option: Reduced Regret

L. Shepp, A.N. Shiryaev (1993)

Mathematical Model

stock price: diffusion process

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad S_0 > 0$$

bank account (money): deterministic process

$$dB_t = rB_t dt \qquad B_0 > 0$$

payoff of the option

$$f_t = e^{-\lambda t} \max\left(\max_{u \le t} S_u, S_0\psi_0\right)$$

where $\lambda > 0$, $\psi_0 > 1$

American type option: $f_{\tau(\omega)}(\omega)$

Question: Fair price of the option?
Optimal stopping time?

Fair price: $C_T^*(f) = B_0 \sup\limits_{0 \le \tau \le T} E^{\mu - r}\left[\dfrac{f_\tau}{B_\tau}\right]$

or $C^*(f) = B_0 \sup\limits_{0 \le \tau < \infty} E^{\mu - r}\left[\dfrac{f_\tau}{B_\tau}\right]$

Solution

Consider process $\psi_t = \dfrac{\max\{\max_{u \le t} S_u, S_0\psi_0\}}{S_t}$

with the following infinitesimal generator

$$L = -r\psi \frac{\partial}{\partial \psi} + \frac{\sigma^2}{2}\psi^2 \frac{\partial^2}{\partial \psi^2}$$

$\longrightarrow$ Stefan problem

$$LV(\psi) = \lambda V(\psi)$$

with boundary condition $V'(1+) = 0$

$$C^*(f) = S_0 \cdot \begin{cases} \dfrac{\widetilde{\psi}}{x_2 - x_1} \left[(x_2 - 1)\left(\dfrac{\psi_0}{\widetilde{\psi}}\right)^{x_1} + (1 - x_1)\left(\dfrac{\psi_0}{\widetilde{\psi}}\right)^{x_2} \right] & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le \psi_0 < \widetilde{\psi} & \\ \psi_0 & \text{for } \psi_0 \ge \widetilde{\psi} \end{cases}$$

where $\widetilde{\psi} = \left| \dfrac{x_2}{x_1} \, \dfrac{x_1 - 1}{x_2 - 1} \right|^{1/(x_2 - x_1)}$

and x_1, x_2 are roots of a quadratic equation

optimal stopping time: $\widetilde{\tau} = \inf \left\{ t \ge 0 \mid \psi_t \ge \widetilde{\psi} \right\}$